

# Sharply 2-transitive groups

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## Abstract

We give an explicit construction of sharply 2-transitive groups with fixed point free involutions and without nontrivial abelian normal subgroup.

## 1 Introduction

The finite sharply 2-transitive groups were classified by Zassenhaus [Z] in the 1930's. They were shown to always contain a regular abelian normal subgroup. It remained an open question whether the same holds for infinite sharply 2-transitive groups. The first examples of sharply 2-transitive groups without abelian normal subgroup were recently constructed in [RST]. In these examples involutions have no fixed points. We here give an alternative approach to such a construction by using partially defined group actions.

## 2 The construction

**Theorem 2.1.** *Let  $G_0$  be a group containing an involution  $t$ . Suppose that  $G_0$  acts on a set  $X$  and satisfies the following:*

1. *no nontrivial element of  $G_0$  fixes more than one element of  $X$  (we say that  $G_0$  is 2-sharp);*
2. *all involutions are conjugate to  $t$ ;*
3.  *$t$  does not fix any element of  $X$ .*

*Then we can extend  $G_0$  to a sharply 2-transitive action of*

$$G = (G_0 *_{\langle t \rangle} (\langle t \rangle \times F(S))) * F(R)$$

*on a suitable set  $Y \supset X$ , where  $F(R), F(S)$  are free groups on disjoint sets  $R, S$  with  $|R|, |S| = \max |G_0|, \aleph_0$ .*

Note that  $G$  does not contain any nontrivial abelian normal subgroup. Hence we obtain:

**Corollary 2.2.** *Any group can be extended to a group acting sharply 2-transitively on some appropriate set without nontrivial abelian normal subgroup.*

*Proof.* By adding a direct factor of order 2 if necessary and iterated HNN-extensions any group can be extended to a group with a unique nontrivial conjugacy class of involutions. Letting this group act regularly on itself by right translation all assumptions of Theorem 2.1 are satisfied.  $\square$

**Definition 2.3.** *A partial action of  $G$  on a set  $X$  consists of an action of  $G_0$  on  $X$  and (injective) partial actions of the generators in  $S \cup R$  such that for  $s \in S, x \in X$  if  $xs$  is defined, then so is  $(xt)s$  and we have  $(xt)s = (xs)t$ .*

Any element of  $G$  can be written as a reduced word in elements of

$$\mathcal{P} = (G_0 \setminus 1) \cup R \cup R^{-1} \cup S \cup S^{-1},$$

where we say that a word is *reduced* if there are no subwords of the form  $g_1 g_2, r^\epsilon r^{-\epsilon}, s^\epsilon s^{-\epsilon}, t s_1^{\pm 1} \dots s_n^{\pm 1} t$  or  $s^\epsilon t s^{-\epsilon}$  for  $g_i \in G_0 \setminus 1, r \in R, s, s_i \in S, \epsilon \in \{1, -1\}$ . It is easy to see that two reduced words represent the same element of  $G$  if and only if they can be transformed into each other by swapping adjacent letters  $t$  and  $s^\epsilon$ .

If  $w = p_1 \dots p_n$  is a word and  $x$  an element of  $X$  we say that  $xw$  is *defined* if for all initial segments of  $w$  the action on  $x$  is defined, i.e. all  $x p_1, (x p_1) p_2, \dots, (\dots (x p_1) \dots) p_n$  are defined and we set  $xw = (\dots (x p_1) \dots) p_n$ . Notice that for elements from  $G_0$  the action on  $X$  is defined everywhere. If  $xw$  is defined and  $w'$  is a reduced word which represents the same element of  $G$  as  $w$ , then  $xw'$  is also defined and we have  $xw = xw'$ . Thus the expression  $xg = y$  makes sense for  $g \in G, x, y \in X$ . Furthermore  $X$  becomes a groupoid with  $\text{hom}(x, y) = \{g \in G \mid xg = y\}$  under the natural map  $\text{hom}(x, y) \times \text{hom}(y, z) \rightarrow \text{hom}(x, z)$ .

If  $G$  acts partially on  $X$ , then there is a canonical partial action on the set of pairs

$$(X)^2 = \{(x, y) \in X^2 \mid x \neq y\}.$$

Notice that since  $t$  does not fix a point, we have  $(x, xt) \in (X)^2$  for all  $x \in X$ . For  $a = (x, y)$  we denote by  $\bar{a}$  the *flip*  $(y, x)$  of  $a$ . If  $ag$  is defined, then so is  $\bar{a}g = \overline{ag}$ .

**Definition 2.4.** *We call a partial action of  $G$  on  $X$  good if for all pairs  $a \in (X)^2$  and  $g \in G$  the following holds:*

1.  $ag = a$  implies  $g = 1$ .
2. If  $ag = \bar{a}$ , then  $g$  is conjugate to  $t$ .
3.  $t$  does not fix an element of  $X$ .

Consider the action of  $G_0$  on  $X$  as a partial action of  $G$  on  $X$ . Then our assumptions on  $G_0$  in Theorem 2.1 translate exactly into saying that  $G$  acts well on  $X$ .

A word in  $\mathcal{P}$  is *cyclically reduced* if every cyclic permutation of  $w$  is reduced. If a word is cyclically reduced, then every reduced word which represents the same element of  $G$  is also cyclically reduced. Thus, to be cyclically reduced is a property of elements of  $G$ . Clearly every element of  $G$  is conjugate to a cyclically reduced one. This shows that in the definition of a good partial action we can restrict ourselves to cyclically reduced elements. Note that the cyclically reduced conjugates of  $t$  are the involutions of  $G_0$ .

**Lemma 2.5** (Extending  $s$ ). *Assume that  $G$  acts well on  $X$  and that for some  $x \in X, s \in S$  and  $\epsilon \in \{1, -1\}$  the expression  $xs^\epsilon$  is not defined (and hence neither is  $xts^\epsilon$ ). Let  $x'G_0 = \{x'g_0 \mid g_0 \in G_0\}$  be a set of new elements on which  $G_0$  acts regularly and extend the partial operation of  $G$  to  $X' = X \cup x'G_0$  by putting  $xs^\epsilon = x'$  and  $(xt)s^\epsilon = x't$ . Then  $G$  acts well on  $X'$ .*

*Proof.* Assume  $\epsilon = 1$ , the other case being entirely similar. Let  $w$  be cyclically reduced and  $aw = a$  in  $X'$ . Then the word  $w$  describes a cycle in  $(X')^2$  containing  $a$ . If the cycle contains pairs from  $X$  only, we are done. If there are two neighbouring pairs in the cycle which do not belong to  $X$ , they must be connected by an element  $g_0 \in G_0 \setminus 1$ . Thus the cycle contains a segment  $b, c'_1, d$  or a segment  $b, c'_1, c'_2, d$  where  $b, d \in X$  and  $c'_i \notin X$ . In the first case we have  $bs = c'_1$ ,  $c'_1s^{-1} = d$  and in the second case  $bs = c'_1$ ,  $c'_1t = c'_2$ ,  $c_2s^{-1} = d$ . In the first case a cyclic permutation of  $w$  contains the subword  $s \cdot s^{-1}$ , in the second case  $s \cdot t \cdot s^{-1}$ . Thus  $w$  is not cyclically reduced, a contradiction.

The proof for  $aw = \bar{a}$  is similar: instead of a cycle such an element  $w$  describes a Moebius strip and we have the additional possibility that  $a = (x', x'i)$  and  $w = i$  for an involution  $i \in G$ .  $\square$

**Lemma 2.6** (Extending  $r$ ). *Assume that  $G$  acts well on  $X$  and that for some  $x \in X, r \in R$  and  $\epsilon \in \{1, -1\}$  the expression  $xr^\epsilon$  is not defined. Choose a set  $x'G_0 = \{x'g_0 \mid g_0 \in G_0\}$  of new elements on which  $G_0$  acts regularly. Extending the partial operation of  $G$  on  $X' = X \cup x'G_0$  by putting  $xr^\epsilon = x'$  yields again a good action of  $G$  on  $X'$ .*

*Proof.* Consider a non-trivial cycle (or Moebius strip) in  $(X)^2$  described by a cyclically reduced word  $w$ . It is easy to see that the cycle (Moebius strip) must either be completely contained in  $(x'G_0)^2$  or completely contained in  $(X)^2$ . In the first case we have a Moebius strip of the form  $(x', x'i)i = (x'i, x')$  for an involution  $i \in G_0$ . The second case cannot occur since  $G$  acts well on  $X$  by assumption.  $\square$

**Lemma 2.7** (Joining  $t$ -pairs). *Assume that  $G$  acts well on  $X$  and let  $a = (x, xt)$  and  $b = (y, yt)$  be pairs for which there is no  $g \in G$  with  $ag = b$ . Let  $s \in S$  be an element which does not yet act anywhere. Extend the action by setting  $as = b$ . Then this action of  $G$  on  $X$  is again good.*

*Proof.* Let  $w$  be a cyclically reduced word with  $cw = c$  for some pair  $c \in (X)^2$ . If  $s$  does not occur in  $w$ , then we have  $w = 1$  since the previous action on  $X$  was good. Hence we may assume that  $w$  contains  $s$ . By cyclically permuting  $w$

and taking inverses we may also assume that  $w = s \cdot w'$  and  $aw = a$  and thus  $bw' = a$ . By assumption on  $a, b$  the subword  $w'$  must contain  $s$ . Hence we may write  $w' = u \cdot s^\epsilon v$  for some subword  $u$  not containing  $s$ . We distinguish two cases:

1.  $\epsilon = 1$ . Then we must have  $bu = a$  or  $bu = \bar{a}$  as  $s$  is only defined on these pairs. Since  $bu = \bar{a}$  implies  $b(ut) = a$  both cases contradict the assumption on  $a, b$ .
2.  $\epsilon = -1$ . Then we have  $bu = b$  or  $bu = \bar{b}$ . If  $bu = b$ , then  $u = 1$  and  $w$  is not reduced. If  $bu = \bar{b} = bt$ , then  $u = t$  and  $w$  contains the subword  $s \cdot t \cdot s^{-1}$ , contradicting the assumption that  $w$  be reduced.

Next we assume that  $w$  is cyclically reduced with  $cw = \bar{c}$  for some pair  $c \in (X)^2$ . If  $w$  does not contain  $s$ , then  $w$  is conjugate to  $t$  since the previous action on  $X$  was good. So we may assume that  $w = s \cdot w'$  and  $aw = \bar{a}$ , i.e.  $bw' = \bar{a}$ . By choice of  $a, b$  we must have  $w'$  containing  $s$  and we see as before that this is impossible.  $\square$

**Lemma 2.8** (Joining other pairs). *Assume that  $G$  acts well on  $X$  and let  $a$  and  $b$  be pairs in  $(X)^2$  such that there is no  $g \in G$  with  $ag = b$  or  $ag = \bar{b}$ . Assume furthermore that there is no  $g$  in  $G$  flipping  $b$  and that the action of  $r \in R$  is not yet defined anywhere. Extending the partial action by setting  $ar = b$  yields again a good action of  $G$  on  $X$ .*

Note that  $a$  may or may not be a  $t$ -pair.

*Proof.* Let  $w$  be cyclically reduced and  $cw = c$  for some pair  $c \in (X)^2$ . If  $r$  does not appear in  $w$ , then we have  $w = 1$  since the previous action on  $X$  is good. Hence we may assume again as before that we have  $w = r \cdot w'$  and  $aw = a$ . Hence  $bw' = a$ . By assumption on  $a, b$ , the word  $w'$  must contain  $r$ . Write  $w' = u \cdot r^\epsilon v$  for some subword  $u$  not containing  $r$ . We distinguish two cases

1.  $\epsilon = 1$ . Then  $bu = a$  or  $bu = \bar{a}$  as  $r$  is only defined there. But this contradicts our choice of  $a, b \in (X)^2$ .
2.  $\epsilon = -1$ . Then we have  $bu = b$  or  $bu = \bar{b}$ . If  $bu = b$ , then we have  $u = 1$  by assumption on the previous action and  $w$  is not reduced. Hence  $bu = \bar{b}$ , contradicting the assumption that no element of  $G$  flips  $b$ .

Now assume that there is some pair  $c$  with  $cw = \bar{c}$ . If  $w$  does not contain  $r$ , then  $w$  is conjugate to  $t$  since the previous action is good. Hence we may again assume that we have  $w = r \cdot w'$  and  $aw = \bar{a}$ , hence  $bw' = \bar{a}$ . By assumption on  $a$  and  $b$ , the word  $w'$  must contain  $r$  and as before we see that this is impossible.  $\square$

**Corollary 2.9.** *Assume that  $G$  acts well on  $X$  with  $|X| \leq \max\{\aleph_0, |G|\}$  and there are sufficiently many elements of  $R$  and  $S$  whose action is not yet defined anywhere. Then we can extend the partial action of  $G$  on  $X$  to a sharply 2-transitive action on some appropriate superset  $Y$ .*

*Proof.* Fix a  $t$ -pair  $a$  in  $X_0$ . Using the previous lemmas we find the set  $Y$  with a 2-sharp action of  $G$  on  $Y$  with the following properties:

1. all  $t$ -pairs are connected to  $a$ ;
2. any pair can be flipped by an element of  $G$ .

The last property can be achieved using Lemma 2.8: if  $b$  cannot be flipped at some stage of the construction, we can connect  $a$  and  $b$  in a later stage. Then  $b$  can be flipped as  $a$  can. Note that Lemma 2.8 is used only for  $t$ -pairs  $a$ .

This easily implies that the action of  $G$  on  $Y$  is sharply 2-transitive: It is left to show that all pairs are connected to  $a$ . Let  $b$  be a pair and  $g \in G$  so that  $bg = \bar{b}$ . Then  $g = hgh^{-1}$  for some  $h \in G$ . This implies  $(bh)t = \bar{b}h$ , so  $bh$  is a  $t$ -pair and whence connected to  $a$  □

This concludes the proof of Theorem 2.1 and its corollary. Note that our construction yields a group action for which no involution has a fixed point.

## References

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